Locus Resulted From Lines Passing Through A Fixed Point And A Closed Curve

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Abstract

The locus problems discussed in this paper are inspired by college entrance exam practice problems from China ([8]). Finding the algebraic equations for the locus that depends on other curves may not be a simple task. However, it will become more intuitive to students if we use a dynamic geometry software (DGS) to trace graphically what the locus may look like first before verifying the equation analytically with a computer algebra system (CAS). In addition to finding the locus for a given problem, we shall generate families of new parametric locus curves from the existing closed curve.

In many places, we first highlight essential algebraic manipulation skills before we explore various scenarios when technological tools are available to learners. The activities described in this paper are accessible to student who have knowledge of parametric equations.

1 Introduction

The problems discussed in this paper are extracted from the article [1]. Finding the equation of a curve defined by the locus of a moving point has been popular and often asked in Gaokao (a college entrance exam) in China. There have been several exploratory activities (see [5] [7] and [9]) derived from Chinese college entrance exam practice problems ([8]). In this paper, we explore the following

Main Problem. If we are given a fixed point A and lines passing through this fixed point to intersect a closed curve at two respective points on the curve, say C and D respectively. The locus E, we are interested in finding, is lying on CD and satisfies $\overrightarrow{ED} = s\overrightarrow{CD}$, where s is a given real number. In addition, we will explore a new family of interesting graphs by varying the parameter s.

Activities explored in this paper can be beneficial to readers who have knowledge in parametric equations. We start with a practice problem originated from ([8]) to initiate our discussions. We demonstrate how a problem can be solved by hand first and also demonstrate those crucial algebraic manipulation skills that are required by high school students from China. In addition to solving simple cases by hand, we typically construct a potential solution geometrically using the trace feature of a DGS such as [2]. Secondly, we seek for a symbolic answer if possible from a symbolic geometry software such as [3]. Finally, we use a CAS (such as [6]) to verify that our analytic solutions are identical to those obtained by using [3]. As we shall see from our discussions, a simple drilled type of examination problem can be turned into many interesting projects for undergraduate students to discover many more unexpected mathematics.

2 The Locus Based On Lines Passing Through A Fixed Point

In the section 2.1, we start with the original problem from [8], which has been modified slightly for a more general setting (see Example 1). Subsequently in the section 2.2, we use technological tools to extend the scenario from a circle to an ellipse. We shall see that knowing the Vieta's Theorem, about how the sum of two roots is related to the coefficients of a quadratic equation, is needed when finding a locus analytically.

2.1 Finding the Locus when the closed curve is a circle

We consider the following problem that is being modified from ([8]).

Example 1 *We are given a fixed circle in black and a fixed point A in the interior of the circle $(x-a)^2+(y-b)^2=r^2$ (see Figure 1(a)). A line passes through A and intersects the circle at C and D respectively, and the point E is the midpoint of CD. Find the locus E.*

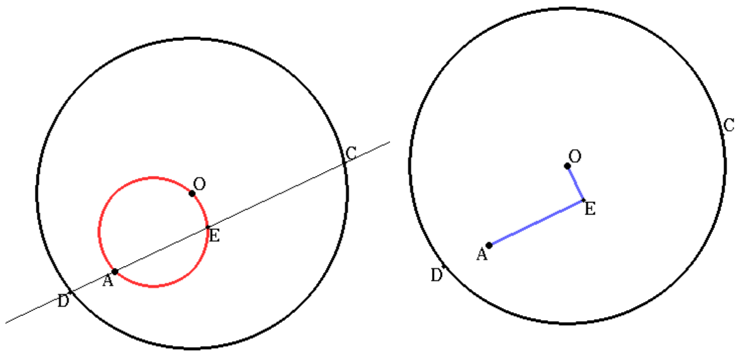

Figure 1(a). Locus and lines passing through a fixed point

Figure 1(b). Locus, circle and perpendicular

We let the fixed point A be (x_0, y_0), and let the line pass through A and intersect the circle at C and D respectively (see Figure 1(a)). We label $E = (x, y)$ as the midpoint of CD. If we denote the center for the circle to be $O = (a, b)$, then using the fact of

$$\overrightarrow{AE} \cdot \overrightarrow{OE} = 0,$$

(see Figure 1(b)), we see

$$(x-x_0)(x-a)+(y-y_0)(y-b)=0. \tag{1}$$

The equation 1 can be reduced to the following:

$$\left(x-\frac{a+x_0}{2}\right)^2+\left(y-\frac{b+y_0}{2}\right)^2=\frac{1}{4}a^2-\frac{1}{2}ax_0+\frac{1}{4}b^2-\frac{1}{2}by_0+\frac{1}{4}x_0^2+\frac{1}{4}y_0^2,$$

which shows that the locus indeed is a circle.

Exploratory Activity. Suppose now we ask the scenario of finding the locus of E satisfying $\overrightarrow{ED}=s\overrightarrow{CD}$, where $s\in(0,1)$, then the problem becomes more complicated, which may not be suitable as an exam question. However, it is a perfect example for students to explore with technological tools. Precisely, suppose C,D are two points on a circle, say $x^2+y^2=r^2$, and the fixed point $A=(u_0,v_0)$ is not the center of the circle. We let E be the point on the line CD passing through A, we want to find the locus of $E=(x,y)$ satisfying $\overrightarrow{ED}=s\overrightarrow{CD}$, where $s\in(0,1)$. We note that the parametric solution for the locus E can be obtained from Geometry Expressions [3] after proper geometric constructions, which we show below: (The exploration template can be found in **[S2]**.)

$$\begin{pmatrix} X(r,s,t,u_0,v_0)=(1-s)\left(\dfrac{-2\left(-v_0+\sin(t)\,|r|\right)\left(u_0\sin(t)\,|r|-v_0\cos(t)\,|r|\right)}{-r^2-u_0^2-v_0^2+2v_0\sin(t)\,|r|+2u_0\cos(t)\,|r|}-\cos(t)\,|r|\right) \\ +s\cos(t)\,|r| \\ Y(r,s,t,u_0,v_0)=(1-s)\left(\dfrac{2\left(-u_0+\cos(t)\,|r|\right)\left(u_0\sin(t)\,|r|-v_0\cos(t)\,|r|\right)}{-r^2-u_0^2-v_0^2+2v_0\sin(t)\,|r|+2u_0\cos(t)\,|r|}-\sin(t)\,|r|\right) \\ +s\sin(t)\,|r| \end{pmatrix}.$$

Next we shall show how this formula is derived analytically, which we provide the corresponding algebraic details in [S1].

Step 1. We label $C=(r\cos t,r\sin t)$, $D=(x_1,y_1)$, and E as (x,y), and observe from $\overrightarrow{DE}=s\overrightarrow{CD}$, where $s\in(0,1)$, that

$$\begin{aligned} x-x_1 &= s\left(x_1-r\cos t\right) \\ x-rs\cos t &= x_1-sx_1 \\ &= x_1\left(1-s\right), \\ y-rs\sin t &= y_1-sy_1 \\ &= y_1\left(1-s\right). \end{aligned}$$

Step 2. Next we note that $D=(x_1,y_1)$ lies on the line equation AC of

$$y-v_0=\left(\frac{r\sin t-v_0}{r\cos t-u_0}\right)\left(x-u_0\right).$$

Since the line AC passes through the fixed point $A=(u_0,v_0)$, we see

$$\begin{aligned} y_1-v_0 &= \left(\frac{r\sin t-v_0}{r\cos t-u_0}\right)\left(x_1-u_0\right), \\ y_1 &= v_0+\left(\frac{r\sin t-v_0}{r\cos t-u_0}\right)\left(x_1-u_0\right). \end{aligned}$$

Step 3. Now, since the line AC intersects the circle of $x^2+y^2=r^2$, we rewrite the equation of the circle by using the line equation AC as follows:

$$x^2+\left(v_0+\left(\frac{r\sin t-v_0}{r\cos t-u_0}\right)(x-u_0)\right)^2=r^2.$$

We label $k=\dfrac{r\sin t-v_0}{r\cos t-u_0}$ and rearrange the quadratic equation as follows:

$$\begin{aligned} k^2u_0^2-2k^2u_0x+k^2x^2-2ku_0v_0+2kv_0x-r^2+v_0^2+x^2 &= 0, \\ \left(1+k^2\right)x^2-\left(2k^2u_0-2kv_0\right)x+\left(k^2u_0^2-2ku_0v_0-r^2+v_0^2\right) &= 0. \end{aligned} \tag{2}$$

Step 4. We solve the quadratic equation involving x and notice that *both* $x_1, r\cos t$ *are two roots of the quadratic equation* 2 above, thus

$$\begin{aligned} x_1+r\cos t &= \frac{2k^2u_0-2kv_0}{1+k^2} \\ x_1 &= \frac{2k^2u_0-2kv_0}{1+k^2}-r\cos t. \end{aligned}$$

Step 5. We write x by using x_1 and substitute this into $x-rs\cos t=x_1(1-s)$ to obtain

$$\begin{aligned} x(r,s,t,u_0,v_0) &= rs\cos t+\left(\frac{2k^2u_0-2kv_0}{1+k^2}-r\cos t\right)(1-s) \\ &= rs\cos t+\left(\frac{2\left(\frac{r\sin t-v_0}{r\cos t-u_0}\right)^2u_0-2\left(\frac{r\sin t-v_0}{r\cos t-u_0}\right)v_0}{1+\left(\frac{r\sin t-v_0}{r\cos t-u_0}\right)^2}-r\cos t\right)(1-s) \\ &= rs\cos t+\left(\frac{2\left(r\sin t-v_0\right)^2u_0-2\left(r\sin t-v_0\right)\left(r\cos t-u_0\right)v_0}{\left(r\cos t-u_0\right)^2+\left(r\sin t-v_0\right)^2}-r\cos t\right)(1-s) \\ &= rs\cos t+\left(\frac{2\left(r\sin t-v_0\right)\left(\left(r\sin t-v_0\right)u_0-\left(r\cos t-u_0\right)v_0\right)}{r^2\cos^2 t-2ru_0\cos t+u_0^2+r^2\sin^2 t-2rv_0\sin t+v_0^2}-r\cos t\right)(1-s) \\ &= rs\cos t+\left(\frac{2\left(r\sin t-v_0\right)\left(ru_0\sin t-rv_0\cos t\right)}{r^2+u_0^2+v_0^2-2ru_0\cos t-2rv_0\sin t}-r\cos t\right)(1-s) \end{aligned}$$

Step 6. We find y_1 and express y accordingly: We substitute x_1 into $y_1=v_0+\left(\frac{r\sin t-v_0}{r\cos t-u_0}\right)(x_1-u_0)$ to get y_1 as follows:

$$y_1=v_0+\left(\frac{r\sin t-v_0}{r\cos t-u_0}\right)\left(\frac{2k^2u_0-2kv_0}{1+k^2}-r\cos t-u_0\right).$$

Next we substitute y_1 into

$$\begin{aligned} y-rs\sin t &= y_1-sy_1 \\ &= y_1(1-s), \end{aligned}$$

to get y as follows:

$$y(r,s,t,u_0,v_0) = rs\sin t + \left(v_0 + \left(\frac{r\sin t - v_0}{r\cos t - u_0}\right)\left(\frac{2k^2u_0 - 2kv_0}{1+k^2} - r\cos t - u_0\right)\right)(1-s)\,.$$

We have used [6] (see [S1]) to verify that, when setting $r > 0$, indeed the solution obtained from Geometry Expressions [3] $[X(r,s,t,u_0,v_0), Y(r,s,t,u_0,v_0)]$ is the same as $[x(r,s,t,u_0,v_0), y(r,s,t,u_0,v_0)]$. We show some screen shots of the locus in red, with respective values of s, in the Figures 2(a)-(c) when using [3] (see [S2]).

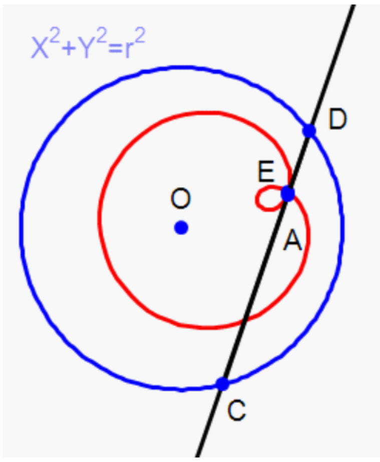

Figure 2(a) Locus and $s = 0.25$

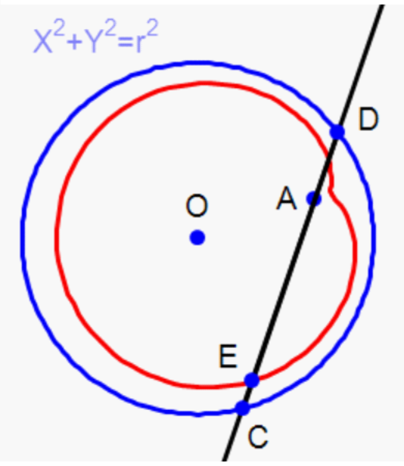

Figure 2(b) Locus and $s = 0.9$

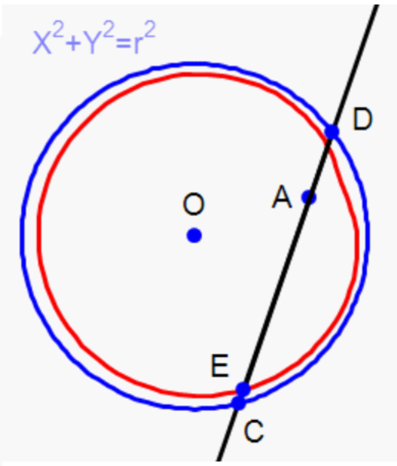

Figure 2(c) Locus and $s = 0.95$

Remarks:

1. We have restricted the fixed point A to be in the interior of the circle, $x^2 + y^2 = r^2$, in our earlier discussion. With a DGS (such as [3]), it is easy to explore scenarios when the point A is outside of a circle. We provide such case in the following screen shot: See Figure 3(a), when

$A = (1.598024, 0.6045678)$ and $r = 2$.

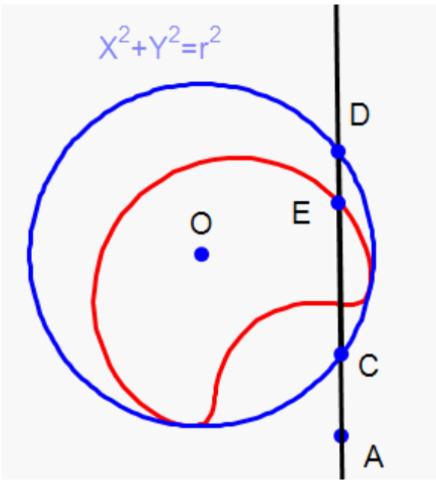

Figure 3(a). A is outside the circle and $s = 0.25$

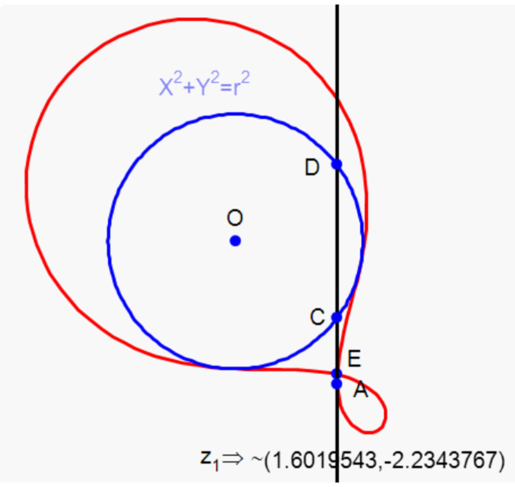

Figure 3(b). A is outside the circle and $s = 1.43$

2. Also, we have restricted s in $(0, 1)$. However, it is easy to see what the locus will look like when $s > 1$. For example, we obtain something interesting if we set $r = 2, s = 1.43$, and the fixed point $(u_0, v_0) = (1.6019543, -2.2343767)$, see Figure 3(b).

2.2 Exploration with a DGS for the case of an ellipse

Next, we naturally replace the circle, $x^2 + y^2 = r^2$, with an ellipse. We discuss the locus when the ratio $s = \frac{1}{2}$ in $\overrightarrow{ED} = s\overrightarrow{CD}$ first in the following Example 2 before exploring other scenarios:

Example 2 *We are given a fixed ellipse in blue and the fixed point A is in the interior of the ellipse. A line passes through A and intersects the ellipse at C and D respectively. If the point E is the midpoint of CD. Then find the locus of E..*

Without loss of generality, we consider the case when the ellipse is in the standard form of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. We let the line pass through the fixed point $A = (u_0, v_0)$ and intersect the ellipse at C and D respectively. In addition, we assume CD is not a vertical line perpendicular to the x axis. If we let the slope CD to be k, then the line equation of $\overleftrightarrow{CD}$ is $y - u_0 = k(x - v_0)$. If we write $C = (x_1, y_1)$ and $D = (x_2, y_2)$, then we use the technique of **difference of squares** to find the equation of the locus. Since C, D are points on the ellipse, we have

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1, \quad (3)$$

$$\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1. \quad (4)$$

We subtract (4) from (3) and see

$$\begin{aligned}
\frac{(x_1 - x_2)(x_1 + x_2)}{a^2} &= -\frac{(y_1 - y_2)(y_1 + y_2)}{b^2} \\
&\implies \frac{y_1 - y_2}{x_1 - x_2} = \frac{-b^2}{a^2}\frac{(x_1 + x_2)}{(y_1 + y_2)} \\
&\implies k = \frac{-b^2}{a^2}\frac{(x_1 + x_2)}{(y_1 + y_2)}.
\end{aligned}$$

If we denote the midpoint E as (X, Y) then $k = \frac{-b^2}{a^2}\frac{X}{Y}$. Since the midpoint E satisfies the line equation $\overleftrightarrow{CD}$, which is passing through the fixed point A, we see $Y - y_0 = \frac{-b^2}{a^2}\frac{X}{Y}(X - x_0)$. Hence, we have $a^2Y(Y - v_0) = -b^2X(X - u_0)$ or $a^2(Y - \frac{v_0}{2})^2 + b^2(X - \frac{u_0}{2})^2 = \dfrac{a^2v_0^2 + b^2u_0^2}{4}$, which yields the followings:

$$\begin{aligned}
\frac{(Y - \frac{v_0}{2})^2}{b^2} + \frac{(X - \frac{u_0}{2})^2}{a^2} &= \frac{a^2v_0^2 + b^2u_0^2}{4a^2b^2}, \\
\frac{(Y - \frac{v_0}{2})^2}{\frac{a^2v_0^2 + b^2u_0^2}{4a^2}} + \frac{(X - \frac{u_0}{2})^2}{\frac{a^2v_0^2 + b^2u_0^2}{4b^2}} &= 1.
\end{aligned}$$

Therefore, the locus E is an ellipse centered at $\left(\frac{u_0}{2}, \frac{v_0}{2}\right)$ whose major and minor lengths are $\frac{\sqrt{a^2v_0^2+b^2u_0^2}}{2b}$ and $\frac{\sqrt{a^2v_0^2+b^2u_0^2}}{2a}$ respectively. We remark that when the point E is the midpoint of CD, finding the locus of E is still manageable when calculated by hand. However, we shall see next that the problem becomes much more challenging when E is not the midpoint.

Exploratory Activities: Suppose we would like to find the locus $E = (X, Y)$ so that $\overrightarrow{CE} = s\overrightarrow{CD}$, where $s \in (0, 1)$. We invite readers to apply the algebraic techniques, which we used for the circle case, to derive the equation of the locus analogously in this case. Consequently, the derived equation of the locus should be identical to the one obtained by Geometry Expressions [3], which we show here:

$$\left(\begin{array}{l}
X = (1-s)\left(\dfrac{2\,(v_0 - \sin(t)\,|b|)\,(-v_0\cos(t)\,|a| + u_0\sin(t)\,|b|)}{b^2\left(-1 - \dfrac{u_0^2}{a^2} - \dfrac{v_0^2}{b^2} + \dfrac{2u_0\cos(t)\,|a|}{a^2} + \dfrac{2v_0\sin(t)\,|b|}{b^2}\right)} - \cos(t)\,|a|\right) + s\cos(t)\,|a| \\
Y = (1-s)\left(\dfrac{2\,(-u_0 + \cos(t)\,|a|)\,(-v_0\cos(t)\,|a| + u_0\sin(t)\,|b|)}{a^2\left(-1 - \dfrac{u_0^2}{a^2} - \dfrac{v_0^2}{b^2} + \dfrac{2u_0\cos(t)\,|a|}{a^2} + \dfrac{2v_0\sin(t)\,|b|}{b^2}\right)} - \sin(t)\,|b|\right) + s\sin(t)\,|b|
\end{array}\right)$$

We use the following screen shots to show the locus in red when $s = 0.3, 0.75$, and 0.8 respectively in Figures 4(a)-(c).

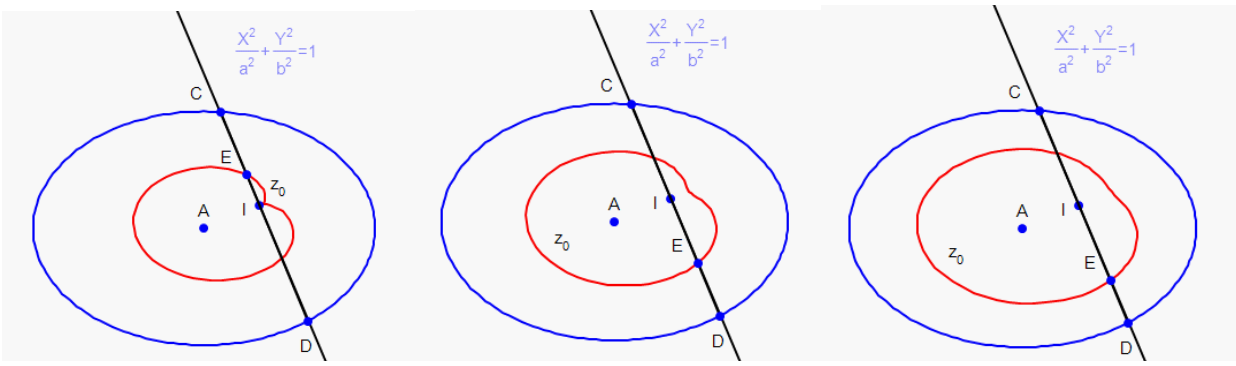

Figure 4(a). Locus, ellipse and $s = 0.3$ Figure 4(b). Locus, ellipse and $s = 0.75$ Figure 4(c). Locus, ellipse and $s = 0.8$

We provide the DGS interactive file in [S3] to allow users to explore if we set the fixed point A outside the given ellipse or $s > 1$.

3 Other Closed Curves When The Fixed Point Is At The Origin

In this section, we discuss a scenario with specific constraints:

1. When the fixed point is at the origin. [It is a much more algebraic intensive problem if we let the fixed point A to be an arbitrary point. To simplify the problem, we set the fixed point to be at the origin.]

2. The concerned closed curve posses *both the polar (or parametric) form and the implicit form of* $f(x, y) = 0$.

Now we replace the ellipse by a cardioid (See Figure 5). Here is the set up of the problem: If A and B are two points on the cardioid and the line $\overleftrightarrow{AB}$ passes through the fixed point $I = (0, 0)$. Find the locus $M = (x, y)$ that satisfies $\overrightarrow{BM} = s\overrightarrow{BA}$, where $s \in (0, 1)$.

We assume the closed curves, considered in this section, posses *both the polar (or parametric) form and the implicit form of* $f(x, y) = 0$. The need of an implicit form for the given curve will be clear later. We shall explore how we use the Vieta's Theorem when finding the locus for this section. We start with the following

Example 3 *We recall the cardioid of* $r = f(t) = 1 - \cos t$ *has the implicit form of* $\left(x^2 + y^2 + x\right)^2 - x^2 - y^2 = 0$. *If* A *and* B *are two points on the cardioid and the line* $\overleftrightarrow{AB}$ *passes through the fixed point* $I = (0, 0)$. *Find the locus* $M = (x, y)$ *that satisfies* $\overrightarrow{BM} = s\overrightarrow{BA}$, *where* $s \in (0, 1)$. *[The locus M is* *shown in red in Figure 5.] The interactive DGS file can be found at [S7].*

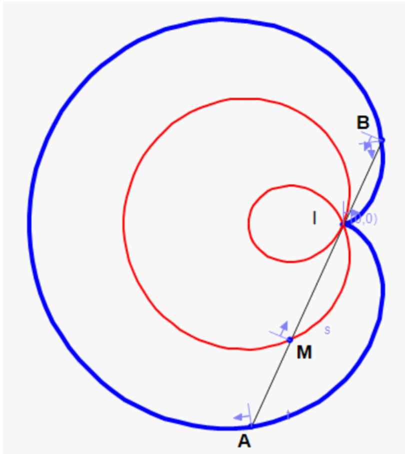

Figure 5. Locus, cardioid and the fixed point at $(0,0)$

We first remark that the following locus can indeed be derived by hand for those students who are not afraid of tedious algebraic manipulations. Next we show how the DGS such as [3] can help students visualize the interesting loci. Finally, we use the CAS such as [6] to show analytically that both answers from [3] and [6] coincide with each other. We provide the interactive CAS file at [S6].

Step 1. We label $A = (f(t)\cos t, f(t)\sin t)$ and $B = (x_1, y_1)$ to be two points on $r = f(t)$. Also we label locus M as (x, y) and observe from $\overrightarrow{BM} = s\overrightarrow{BA}$ that

$$\begin{aligned} \begin{bmatrix} x - x_1 \\ y - y_1 \end{bmatrix} &= s \begin{bmatrix} f(t)\cos t - x_1 \\ f(t)\sin t - y_1 \end{bmatrix} \\ \begin{bmatrix} x \\ y \end{bmatrix} &= (1-s)\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + s\begin{bmatrix} f(t)\cos t \\ f(t)\sin t \end{bmatrix} \end{aligned} \tag{5}$$

Step 2. Since $\overleftrightarrow{AB}$ passes through the fixed point I at the origin, we write the line of $\overleftrightarrow{AB}$ as $y = mx$.

We shall see the need of knowing the implicit equation, when finding the locus for this problem, in the following step:

Step 3. Now, we plug $y = mx$ into the implicit equation of the cardioid, $(x^2 + y^2 + x)^2 - x^2 - y^2 = 0$ and obtain the following:

$$\begin{aligned} \left(x^2 + m^2x^2 + x\right)^2 - x^2 - m^2x^2 &= 0 \\ x^2\left(m^4x^2 + 2m^2x^2 + 2m^2x - m^2 + x^2 + 2x\right) &= 0 \end{aligned}$$

If $x = 0$ then $B = (0, 0)$ which implies $B = I$, the problem becomes a simple exercise to explore, and we leave it to readers to verify.

If $x \neq 0$, then

$$x^2\left(m^4 + 2m^2 + 1\right) + \left(2m^2 + 2\right)x - m^2 = 0, \tag{6}$$

and we consider the discriminant of the quadratic equation in x (6) as follows:

$$D = \left(2m^2+2\right)^2 + 4\left(m^4+2m^2+1\right)m^2 > 0.$$

Since $D > 0$, the two roots x_1^* and x_2^* from (6) should satisfy $x_1^* + x_2^* = \frac{-\left(2m^2+2\right)}{(m^4+2m^2+1)}$ by Vieta's Theorem. We write

$$\begin{aligned}
x_1^* &= \frac{-\left(2m^2+2\right)}{\left(m^4+2m^2+1\right)} - f(t)\cos t \\
&= \frac{-\left(2\tan^2 t+2\right)}{\left(\tan^4 t+2\tan^2 t+1\right)} - f(t)\cos t \\
y_1^* &= (\tan t)\, x_1^* \\
&= (\tan t)\left(\frac{-\left(2\tan^2 t+2\right)}{\left(\tan^4 t+2\tan^2 t+1\right)} - f(t)\cos t\right)
\end{aligned}$$

Step 4. We write x by using x_1^* in (5), in other words, we have $x - f(t)s\cos t = x_1 - sx_1$ $= x_1(1-s)$, which implies the following:

$$\begin{aligned}
x(s,t) &= f(t)s\cos t + x_1^*(1-s) \\
&= s(1-\cos t)\cos t + (1-s)\left(\frac{-\left(2\tan^2 t+2\right)}{\left(\tan^4 t+2\tan^2 t+1\right)} - f(t)\cos t\right) \\
&= s(1-\cos t)\cos t + (1-s)\left(\frac{-\left(2\tan^2 t+2\right)}{\left(\tan^4 t+2\tan^2 t+1\right)} - (1-\cos t)\cos t\right)
\end{aligned}$$

Step 5. We use y_1^* to find y (5) In other words, we have

$$\begin{aligned}
y(s,t) &= sf(t)\sin t + y_1^*(1-s) \\
&= s(1-\cos t)\sin t + (1-s)\left(\tan t\left(\frac{-\left(2\tan^2 t+2\right)}{\left(\tan^4 t+2\tan^2 t+1\right)} - f(t)\cos t\right)\right) \\
&= s(1-\cos t)\sin t + (1-s)\left(\tan t\left(\frac{-\left(2\tan^2 t+2\right)}{\left(\tan^4 t+2\tan^2 t+1\right)} - (1-\cos t)\cos t\right)\right)
\end{aligned}$$

Step 6. We remark that the output of the parametric equation for the locus from [3] is shown below

$$\left(\begin{array}{l} X(s,t) = (-1+2s)\cos(t) - \cos(t)^2 \\ Y(s,t) = (-1+2s-\cos(t))\sin(t) \end{array}\right).$$

After using `simplify` command in [6], we see $x(s,t) = X(s,t)$ and $y(s,t) = Y(s,t)$. We show various screen shots of the locus obtained from the CAS Maple [6], which correspond to their respective value s below in Figures 6(a)-(d).

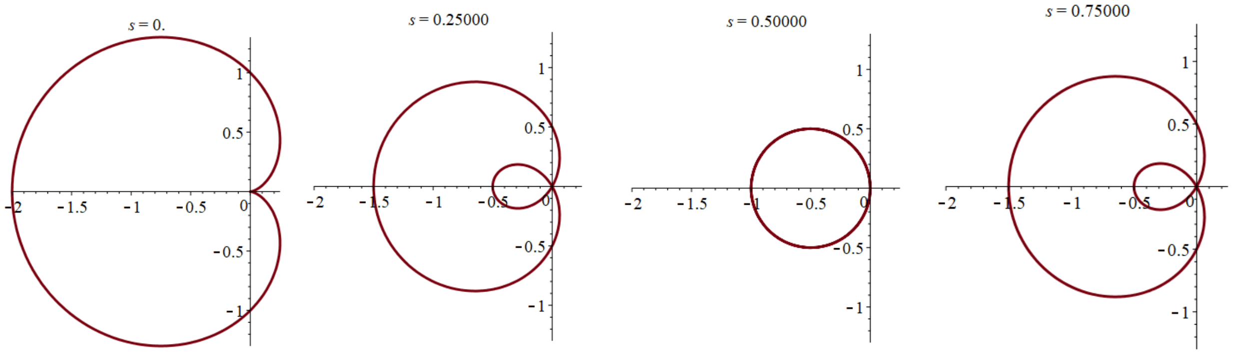

Figure 6(a) Maple plot with $s = 0$ | Figure 6(b) Maple plot with $s = 0.25$ | Figure 6(c) Maple plot with $s = 0.5$ | Figure 6(d) Maple plot with $s = 0.75$

We explore another scenario as follows:

Example 4 *We consider the polar equation of* $r = 3 + 7\sin 3t$, $t \in [0, 2\pi]$, *which can be represented (with the help of [3]) by the implicit equation of* $-9x^6 + x^8 - 42x^6y + 414x^4y^2 + 4x^6y^2 - 70x^4y^3 - 321x^2y^4 + 6x^4y^4 - 14x^2y^5 + 40y^6 + 4x^2y^6 + 14y^7 + y^8 = 0$*. If the fixed point* I *is at the origin and the line* $\overleftrightarrow{AB}$ *passes through the fixed point* I*. Find the locus* $M = (X, Y)$ *satisfying* $\overrightarrow{BM} = s\overrightarrow{BA}$*, where* $s \in (0, 1)$.

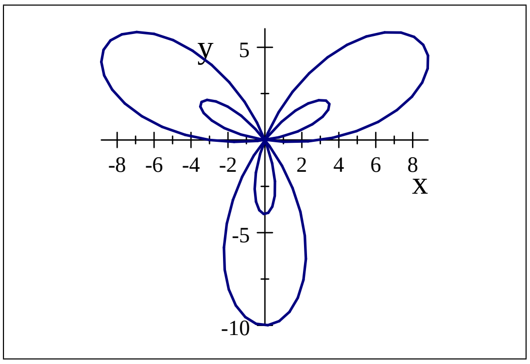

Figure 7. Graph of
$r = 3 + 7\sin 3t, t \in [0, 2\pi]$

We provide the CAS and DGS interactive files at [S6 and S7] respectively.

Step 1. We set $f(t) = 3 + 7\sin 3t$ and label $A = (f(t)\cos t, f(t)\sin t)$ and $B = (x_1, y_1)$ to be two points on $r = f(t)$. Also we label locus M as (x, y) and observe from $\overrightarrow{BM} = s\overrightarrow{BA}$ that

$$\begin{aligned} \begin{bmatrix} x - x_1 \\ y - y_1 \end{bmatrix} &= s \begin{bmatrix} f(t)\cos t - x_1 \\ f(t)\sin t - y_1 \end{bmatrix} \\ x - f(t)s\cos t &= x_1 - sx_1 \\ &= x_1 (1 - s), \\ y - f(t)s\sin t &= y_1 - sy_1 \\ &= y_1 (1 - s). \end{aligned} \tag{7}$$

Step 2. Since $\overleftrightarrow{AB}$ passes through the fixed point I at the origin, we write the line of $\overleftrightarrow{AB}$ as $y = mx$.

We shall see the need of knowing the implicit equation, when finding the locus for this problem, in the following step:

Step 3. Now, we plug $y = mx$ into the implicit equation of the polar equation $r = f(t)$ and obtain $x^6(x^2m^8 + 4x^2m^6 + 14xm^7 + 6x^2m^4 - 14xm^5 + 40m^6 + 4x^2m^2 - 70xm^3 - 321m^4 + x^2 - 42xm + 414m^2 - 9) = 0$

If $x = 0$ then $B = (0, 0)$ which implies $B = I$, the problem becomes a simple exercise to explore, and we leave it to readers to verify.

If $x \neq 0$, then

$$x^2 \left(m^2 + 1\right)^4 + x \left(14m \left(m^2 - 3\right) \left(m^2 + 1\right)^2\right) + \left(414m^2 - 321m^4 + 40m^6 - 9\right) = 0. \tag{8}$$

We consider the discriminant of the quadratic equation in x (8) and use CAS [6] to simplify as follows:

$$D = 36 \left(m^2 + 1\right)^7 > 0.$$

Since $D > 0$, we note two roots x_1^* and x_2^* from (8) satisfying

$$x_1^* + x_2^* = -\frac{14\left(\tan^2 t - 3\right)\tan t}{\left(\tan^2 t + 1\right)^2}$$

by Vieta's Theorem. We write

$$\begin{aligned} x_1^* &= -\frac{14\left(\tan^2 t - 3\right)\tan t}{\left(\tan^2 t + 1\right)^2} - f(t)\cos t \\ y_1^* &= (\tan t)\, x_1^* \\ &= (\tan t)\left(-\frac{14\left(\tan^2 t - 3\right)\tan t}{\left(\tan^2 t + 1\right)^2} - f(t)\cos t\right) \end{aligned}$$

Step 4. We write x_1 by using x_1^* in (5). In other words, we have $x - f(t)s\cos t = x_1 - sx_1$ $= x_1 (1 - s)$, which implies the following:

$$\begin{aligned} x(s,t) &= f(t)s\cos t + x_1^*(1 - s) \\ &= s\,(3 + 7\sin 3t)\cos t + (1 - s)\left(-\frac{14\left(\tan^2 t - 3\right)\tan t}{\left(\tan^2 t + 1\right)^2} - (3 + 7\sin 3t)\cos t\right) \end{aligned}$$

Step 5. We use y_1^* to find $y\ (5)$. In other words, we have

$$\begin{aligned} y(s,t) &= sf(t)\sin t + y_1^*\,(1-s) \\ &= s(3+7\sin 3t)\sin t + (1-s)\left((\tan t)\left(-\frac{14\,(\tan^2 t - 3)\tan t}{(\tan^2 t+1)^2} - (3+7\sin 3t)\cos t\right)\right) \end{aligned}$$

We remark that the output of the parametric equation for the locus from [3] is shown to be identical, after using `simplify` command in [6], with $[x(s,t), y(s,t)]$. We show various screen shots of the locus obtained from the CAS Maple [6], which corresponding to their respective value s in Figures 8(a)-(d). Incidentally, we discovered a way of constructing a three-leaf rose along the process of exploring locus in this case (see Figure 8(c)).

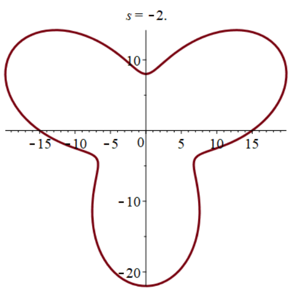

Figure 8(a). Locus when $s=-2$

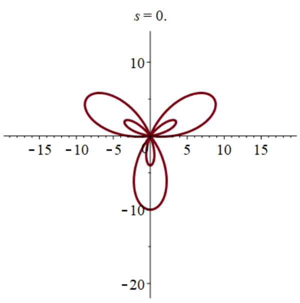

Figure 8(b). Locus when $s=0$

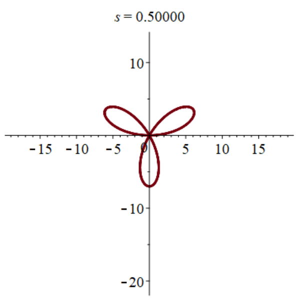

Figure 8(c). Locus when $s=0.5$

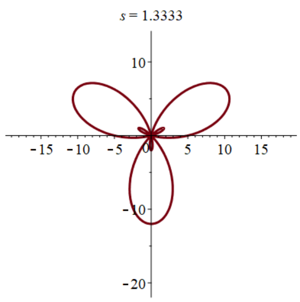

Figure 8(d). Locus when $s=1.3333$

We invite readers to explore the following analogous scenario:

Exercise 5 *We are given the following curve* $r=\sin t-\sin 2t, t\in[0,2\pi]$, *which looks like a butterfly- see Figure 9.*

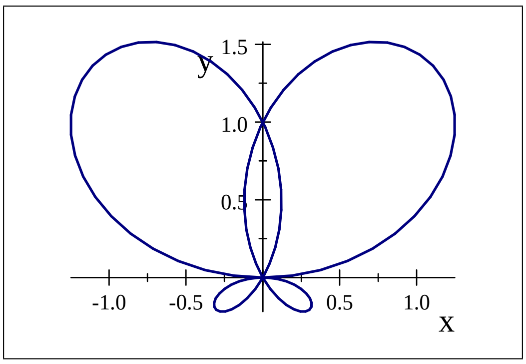

Figure 9. Graph of $r=\sin t-\sin 2t, t\in[0,2\pi]$

The corresponding implicit equation is given as $x^6-2x^4y-3x^2y^2+3x^4y^2-4x^2y^3+y^4+3x^2y^4-2y^5+y^6=0$. *If the fixed point* I *is at the origin and the line* $\overleftrightarrow{AB}$ *passes through the fixed point* I. *Find the locus* $M=(X,Y)$ *satisfying* $\overrightarrow{BM}=s\overrightarrow{BA}$, *where* $s\in(0,1)$. *The interactive CAS and DGS files can be found at [S8] and [S9] respectively.*

We encourage readers to explore using the algebraic method mentioned in the preceding example and verify that the parametric equation for the locus to be shown as follows:

$$\begin{aligned} X(s,t) &= (1-s)\left(\frac{2(2\tan t^3+\tan t^5+\tan t)}{(3\tan t^2+3\tan t^4+\tan t^6+1)}-f(t)\cos t\right) \\ Y(s,t) &= sf(t)\sin t+(1-s)\left((\tan t)\left(\frac{2(2\tan t^3+\tan t^5+\tan t)}{(3\tan t^2+3\tan t^4+\tan t^6+1)}\right)\right), \end{aligned}$$

where $f(t)=\sin t-\sin 2t, t\in[0,2\pi]$.

The following shows various screen shots for the corresponding ratios s in Figures 10(a)-(d) respectively. More exploratory details can be found from [S10].

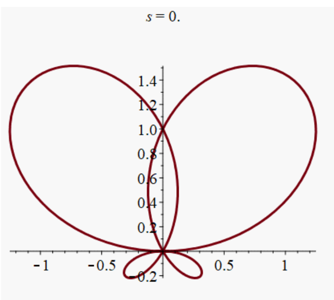

Figure 10(a). Locus when $s=0$

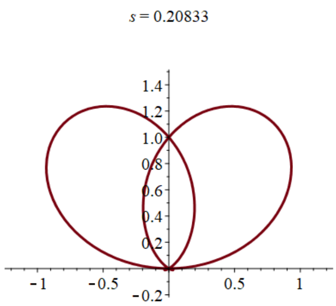

Figure 10(b). Locus when $s=0.20833$

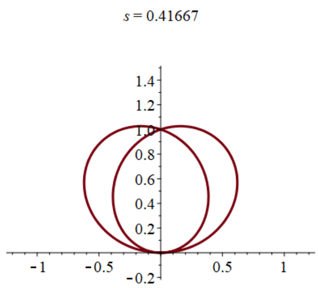

Figure 10(c). Locus when $s=0.41667$

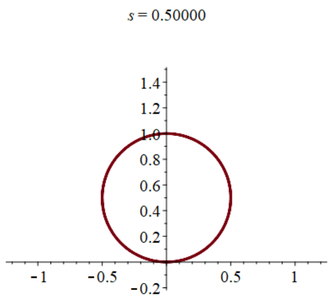

Figure 10(d). Locus when $s=0.5$

Finally, we use the following example to caution the readers that the implicit form for a curve might generate an extraneous solution in some cases.

Example 6 *We consider the polar equation of* $r=2-\sin 2t$*, which can be represented by the implicit equation of* $8x^3y+8xy^4-12x^2y^2+3x^2y^4+3x^4y^2-4x^4+x^6-4y^4+y^6=0$ *(with the help of Geometry Expressions). If the fixed point* I *is at the origin and the line* $\overleftrightarrow{AB}$ *passes through the fixed point* I*. Find the locus* $M=(X,Y)$ *satisfying* $\overrightarrow{BM}=s\overrightarrow{BA}$*, where* $s\in(0,1)$*.*

It is easy to see that the implicit equation does include the point of $(0,0)$ but $r = 2 - \sin 2t$ does not (see the curve of $r = 2 - \sin 2t$ in Figure 11).

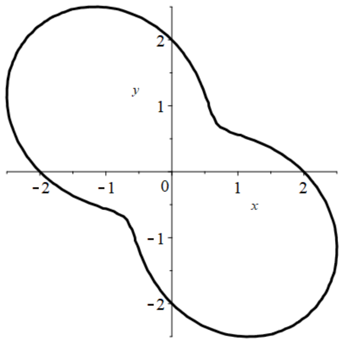

Figure 11. Plot of
$r = 2 - \sin 2t$

Furthermore, we observe from the curve of $r = f(t)$ that it is symmetric to $(0,0)$. Therefore, if $\overleftrightarrow{AB}$ is a line passing through the fixed point at the origin, and $A = (f(t)\cos t, f(t)\sin t)$, then $B = (-x_1, -y_1) = (f(t)\cos t, f(t)\sin t)$. We follow the standard procedure to find the locus $M = (x, y)$ as follows:

Step 1. We label the locus M as (X, Y), and observe from $\overrightarrow{BM} = s\overrightarrow{BA}$, where $s \in (0,1)$, that

$$\begin{aligned} \begin{bmatrix} X - x_1 \\ Y - y_1 \end{bmatrix} &= s \begin{bmatrix} f(t)\cos t - x_1 \\ f(t)\sin t - y_1 \end{bmatrix} \\ X - f(t)s\cos t &= x_1 - sx_1 \\ &= x_1 (1-s), \\ Y - f(t)s\sin t &= y_1 - sy_1 \\ &= y_1 (1-s). \end{aligned}$$

Step 2. We write X by using x_1, which yields, $x - f(t)s\cos t = x_1 - sx_1 = x_1 (1-s)$. We substitute this into the following:

$$\begin{aligned} X(s,t) &= sf(t)\cos t + x_1(1-s) \\ &= sf(t)\cos t - (f(t)\cos t)(1-s) \end{aligned}$$

Step 3. We use y_1 and $Y - f(t)s\sin t = y_1 - sy_1 = y_1(1-s)$ to find y.

$$\begin{aligned} Y(s,t) &= sf(t)\sin t + y_1 (1-s) \\ &= sf(t)\sin t - (f(t)\sin t)(1-s), \end{aligned}$$

where $f(t) = 2 - \sin 2t, t \in [0, 2\pi]$. It is not surprising to imagine that the locus will be of different sizes (expansion or shrinking depending s) from the original polar curve, which we encourage readers to verify on their own by exploratory files in [S10 and S11].

Remarks:

For the locus problems we discussed in Examples 3,4 and 6, and Exercise 5, we have assumed that the implicit equation for a given closed curve is known. This will allow us to apply the Vieta's Theorem to find the locus mentioned in the *Main Problem*. We invite readers to investigate how to categorize those closed curves, in implicit equation forms, where algebraic method by Vieta's Theorem will work or will not work when finding the locus.

4 Conclusion

It is clear that technological tools provide us with many crucial intuitions before we attempt to find rigorous analytical solutions. Here we have gained geometric intuitions while using a DGS such as [2] or [3]. In the meantime, we use a CAS such as [6], for verifying that our analytical solutions are consistent with our initial intuitions. The complexity level of the problems we posed vary from the simple to the difficult. Many of our solutions are accessible to readers who have knowledge in parametric equations. In particular, author believes that the problems mentioned in this paper can be excellent projects for professional trainings for future math teachers and students from university levels.

Evolving technological tools definitely have made mathematics fun and accessible on one hand, but they also allow the exploration of more challenging and theoretical mathematics. We hope that when mathematics is made more accessible to students, it is possible more students will be inspired to investigate problems ranging from the simple to the more challenging. We do not expect that exam-oriented curricula will change in the short term. However, encouraging a greater interest in mathematics for students, and in particular providing them with the technological tools to solve challenging and intricate problems beyond the reach of pencil-and-paper, is an important step for cultivating creativity and innovation.

5 Acknowledgements

The author would like to thank anonymous reviewer for proving many instructive comments to this paper. The author would also express sincere gratitude to Dao Hoang and Phil Todd (founder of [3]) for creating instrumental dynamic figures using Geometry Expressions ([3]). Finally, the author also would like to thank Qiuxia Li for providing the proofs of some examples and numerous inspiring discussions about the level of high school math content knowledge from her teaching experiences in China.

6 Supplementary Electronic Materials

[S1] Maple worksheet for Example 1.

[S2] Geometry Expressions worksheet Example 1.

[S3] Geometry Expressions worksheet Example 2.

[S4] Maple worksheet for Example 3.

[S5] Geometry Expressions worksheet Example 3.

[S6] Maple worksheet for Example 4.

[S7] Geometry Expressions worksheet Example 4.

[S8] Maple worksheet for Exercise 5.

[S9] Geometry Expressions worksheet Exercise 5.

[S10] Maple worksheet for Example 6.

[S11] Geometry Expressions worksheet for Example 6.

References

[1] Yang, W.C. (2018) Strong Algebraic Manipulation Skills Are Not Adequate For Cultivating Creativity And Innovation, *Proceedings of the ATCM 2018*, November 20-24, ISBN 978-0-9972807-2-2, 2018.

[2] ClassPad Manager, a product of CASIO, https://edu.casio.com/freetrial/freetrial_list.php

[3] Geometry Expressions, see http://www.geometryexpressions.com/.

[4] Geometry Interactive Mathematical Arts (GInMA): A Dynamic Geometry System, see http://deoma-cmd.ru/en/Products/Geometry/GInMA.aspx.

[5] Yang, W.-C., (2017) From Static Locus Problems to Exploring Mathematics with Technological Tools, *The Electronic Journal of Mathematics and Technology*, Volume 11, Number 2, pp. 67-88, 2017.

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[7] McAndrew, A., Yang, W.-C. (2016). Locus and Optimization Problems in Lower and Higher Dimensions. The Electronic Journal of Mathematics and Technology, 10(2), 69-83, https://php.radford.edu/~ejmt/deliveryBoy.php?paper=eJMT_v10n2p1.

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